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RATE-INDEPENDENT PROCESSES WITH LINEAR GROWTH ENERGIES AND TIME-DEPENDENT BOUNDARY CONDITIONS

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Abstract. A rate-independent evolution problem is considered for which the stored energy density depends on the gradient of the displacement. The stored energy density does not have to be quasiconvex and is assumed to exhibit linear growth at infinity; no further assumptions are made on the behaviour at infinity. We analyse an evolutionary process with positively 1-homogeneous dissipation and time-dependent Dirichlet boundary conditions.

1. **Introduction.** In this contribution, we analyse a rate-independent mesoscopic process governed by time-dependent Dirichlet boundary conditions. A characteristic feature of the problem is that the stored energy has linear growth at infinity. A similar problem for fixed Dirichlet data, but a time-dependent applied force has been previously considered by the authors [9]. That formulation, however, requires a restriction on the norm of the applied force. This difficulty vanishes in the case under consideration.

The rate-independent process we consider models plastic behaviour of a solid. A sketch of the motivation is as follows (see also [9]). Crystalline materials can often be characterised via energy minimisation; for plastically deformed crystals, Ortiz and Repetto [13] provide a setting in which dislocation structures can be described by a nonconvex minimisation problem. The nature of this variational model is incremental, to reflect the irreversible nature of plastic deformations [13]. We account for these phenomena with a phenomenological dissipation functional. As discussed elsewhere [9], one is led to an energy that depends on a strain tensor and has linear growth at infinity. One important feature of the analysis is that we do not work in BV, since the time-dependent boundary data require continuity of the trace, while the variational arguments build on compactness. To get this combination, we use a fine extension developed by J. Souček, see Subsection 1.2.

The motivation for the analysis of linearly growing energies stem from applications in plasticity. In particular, Conti and Ortiz [2] derive an energy that is

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linear except for the trace part. They consider single-crystal plasticity in limiting case of infinite hardening. If $u\colon \Omega \to \mathbb{R}^3$ is the displacement, set $\beta^{\mathrm{sym}} := e(u) := \frac{1}{2}\left(Du + Du^T\right)$. We write the plastic strain in single crystals for monotonic deformations as $e^p(u) := \frac{1}{2}\left(\beta^p + \beta^{pT}\right)$, where

$$\beta^p(\gamma) = \sum_{j=1}^{J} \gamma_j s_j \otimes m_j,$$

with γ_j being the *slip strain*, s_j the *slip direction* and m_j the *plane normal*. If one assumes infinite latent hardening and no self-hardening, then one is led to a microscopic energy W that is linear along single-slip orbits. For the macroscopic energy, Conti and Ortiz have shown that the convex envelope in this situation has linear growth on traceless symmetric matrices, and quadratic on trace part

$$c\left(|\beta^{\mathrm{sym}}| + |\mathrm{Tr}(\beta)|^2 - 1\right) \le W^{**}(\beta) \le C\left(|\beta^{\mathrm{sym}}| + |\mathrm{Tr}(\beta)|^2 + 1\right)$$

Thus, the macroscopic energy is linear except for the trace. We focus here on this linear growth behaviour alone for the sake of clarity of the exposition; the inclusion of a quadratically growing energy of the trace is a technical issue we do want to discuss here.

This article is organised as follows. Subsection 1.1 settles the notation; a short synopsis of Young measures and DiPerna-Majda measures is given in Appendix A. We refer the reader to [9] for a similar but slightly more comprehensive overview. Section 2 describes the evolutionary problem with time-dependent boundary conditions; Section 3 states the required assumptions and Section 4 gives the (constructive) existence proof.

1.1. **Basic notation.** Let X be a topological space. We denote the space of real-valued continuous functions in X by $C(X)$. If X is a locally compact space then $C_0(X)$ denotes the closure of the subspace of functions with the compact support in $C(X)$. We write $(X, \mathcal{M}, \mu)$ for a measurable space with σ-algebra $\mathcal{M}$. For simplicity, μ is omitted in the notation if $X \subset \mathbb{R}^n$ is open and μ is the n-dimensional Lebesgue measure. We recall that the *support* of a Borel measure μ is the complement of the largest open set N with $\mu(N) = 0$.

If X is a locally compact Hausdorff space, we write $M(X)$ for the set of (signed) Radon measures with finite mass supported on X; $M^+(X)$ stands for the cone of non-negative Radon measures; $\mathrm{Prob}(X)$ is the set of probability measures. The Jordan decomposition for signed measures $\mu = \mu^+ - \mu^-$ gives rise to the total variation $|\mu| := \mu^+ + \mu^-$. The set $M(X)$ is a Banach space when endowed with the total variation $\|\mu\| := |\mu|(X)$ as a norm. By the Riesz Representation Theorem, the dual space to $C_0(X)$, $C_0(X)'$, is isometrically isomorphic with $M(X)$. The weak-$\star$ topology on $M(X)$ is defined by this duality and weak-$\star$ convergence is denoted $u_k \overset{\star}{\rightharpoonup} u$. Finally, if X is compact then the dual space to $C(X)$, $C(X)'$, is isometrically isomorphic with $M(X)$.

The usual Lebesgue space of p-integrable functions is denoted by $L^p(X, \mu)$. Again, we suppress μ from the notation if it is the Lebesgue measure. The notation $\langle \mu, f\rangle := \int_\Omega f(x)\, \mu(\mathrm{d}x) = \int_\Omega f(x)\,\mathrm{d}x$ is used interchangeably. Further, $W^{1,1}_{u_D}(\Omega; \mathbb{R}^m)$ stands for the set of functions $u \in W^{1,1}(\Omega; \mathbb{R}^m)$ with $u = u_D$ on Γ_D. Here $\Gamma_D \subset \partial\Omega$ has a positive $n-1$ dimensional Hausdorff measure and $u_D \in W^{1,1}(\Omega; \mathbb{R}^m)$ is given. Throughout the article, $\Omega \subset \mathbb{R}^n$ is always a bounded domain with smooth boundary. Weak convergence respectively strong convergence is expressed as $u_k \rightharpoonup u$ respectively $u_n \to u$ as usual. We follow the convention of writing C for a generic constant, whose value may change from line to line.

1.2. **Fine extensions of $W^{1,1}(\Omega;\mathbb{R}^m)$.** It is well known that $W^{1,1}(\Omega;\mathbb{R}^m)$ is non-reflexive, that is, a bounded sequence does not necessarily contain a subsequence with a weak limit in $W^{1,1}(\Omega;\mathbb{R}^m)$. Hence, one often looks for an extension of $W^{1,1}(\Omega;\mathbb{R}^m)$. Instead of the usual space of functions of bounded variations, we will work with the so-called Souček space [15]; we denote it by $W^{1,\mu}\left(\bar\Omega;\mathbb{R}^m\right)$. This extension consists of functions in $L^1(\Omega;\mathbb{R}^m)$ whose gradient is a measure on $\bar\Omega$ (see [9], where a similar but more extensive summary is given). The precise formulation is as follows. Let

$$\begin{aligned} W^{1,\mu}\left(\bar\Omega;\mathbb{R}^m\right) = \Big\{ &\left(u,\bar Du\right) \in L^1(\Omega;\mathbb{R}^m) \times M\left(\bar\Omega\right); \\ &\text{there exists } \{u_k\}_{k\in\mathbb{N}} \subset W^{1,1}(\Omega;\mathbb{R}^m) \text{ such that} \\ &u_k \to u \text{ in } L^1(\Omega;\mathbb{R}^m) \text{ and } \nabla u_k \to \bar Du \text{ weakly}\star \text{ in } M\left(\bar\Omega;\mathbb{R}^{m\times n}\right) \Big\}. \end{aligned}$$

It is known [15] that $W^{1,\mu}\left(\bar\Omega;\mathbb{R}^m\right)$ is a Banach space if equipped with the norm

$$\left\|\left(u,\bar Du\right)\right\|_{W^{1,\mu}\left(\bar\Omega;\mathbb{R}^m\right)} = \|u\|_{L^1(\Omega;\mathbb{R}^m)} + \left\|\bar Du\right\|_{M\left(\bar\Omega;\mathbb{R}^{m\times n}\right)}.$$

The weak$\star$ convergence in $W^{1,\mu}\left(\bar\Omega;\mathbb{R}^m\right)$ is defined analogously to $BV(\Omega;\mathbb{R}^m)$; the precise formulation can be found in the literature [15, 9]. Moreover, as shown in [15, Theorem 1 (iii)], if $\left(u,\bar Du\right) \in W^{1,\mu}\left(\bar\Omega;\mathbb{R}^m\right)$, then there is a unique measure $\bar T\left(u,\bar Du\right) \in M(\partial\Omega;\mathbb{R}^m)$ such that

$$\int_{\partial\Omega} (\varphi\cdot\nu)\left(\bar T\left(u^j,\bar Du^j\right)\right)(\mathrm{d}A) = \int_\Omega u^j(x)\,\operatorname{div}\,\varphi(x)\,\mathrm{d}x + \int_{\bar\Omega} \varphi\cdot\left(\bar Du^j\right)(\mathrm{d}x) \qquad (1)$$

for all $\varphi \in C^1\left(\bar\Omega;\mathbb{R}^n\right)$ and all $1 \le j \le m$, where ν is the outward pointing normal. The measure

$$\bar T\left(u,\bar Du\right) = \left(\bar T\left(u^1,\bar Du^1\right),\ldots,\bar T\left(u^m,\bar Du^m\right)\right)$$

is called the *trace* of $\left(u,\bar Du\right)$. Here, the measure $\bar Du^j$ denotes the jth row of the matrix-valued measure $\bar Du$. We now quote the key results which provide a mathematical justification for working in $W^{1,\mu}\left(\bar\Omega;\mathbb{R}^m\right)$: compactness holds as for BV in the weak topology, but in addition the trace operator is continuous in suitable topologies. This enables us to impose Dirichlet boundary data, which would pose a challenge in the conventional setting of BV. While this is a mathematical justification, the question whether $W^{1,\mu}\left(\bar\Omega;\mathbb{R}^m\right)$ is also the appropriate space in the sense of mechanics, giving agreement with experimental observations, is open.

The operator $W^{1,\mu}\left(\bar\Omega;\mathbb{R}^m\right) \to M(\partial\Omega;\mathbb{R}^m)$ given by $(u,Du) \mapsto \bar Tu$ is (weak$\star$, weak$\star$) continuous [15, Theorem 2 (ii)]. Finally, balls in $W^{1,\mu}\left(\bar\Omega;\mathbb{R}^m\right)$ are weakly$\star$ compact, which can be seen as in [15, Theorem 6]. The following Poincaré-type inequality has been proved recently [9].

Lemma 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, with $\partial\Omega$ belonging to class C^1. Let $\Gamma_D \subset \partial\Omega$ be relatively open and of positive $(n-1)$- dimensional Lebesgue measure; suppose further that $z \in M(\Gamma_D;\mathbb{R}^m)$. Then there is $C > 0$ such that the estimate*

$$\|u\|_{L^1(\Omega;\mathbb{R}^m)} \le C\left(\left\|\bar Du\right\|_{M\left(\bar\Omega;\mathbb{R}^{m\times n}\right)} + \|z\|_{M(\Gamma_D;\mathbb{R}^m)}\right) \qquad (2)$$

holds for all $\left(u,\bar Du\right) \in W^{1,\mu}\left(\bar\Omega;\mathbb{R}^m\right)$ with $\bar T\left(u,\bar Du\right) = z$ on Γ_D .

2. **Rate-independent evolution with linearly growing energy and time-dependent boundary conditions.** We now have the ingredients to start the analysis of a rate-independent mesoscopic process governed by time-dependent Dirichlet boundary conditions. The focus is on a relaxed formulation of a problem with linear growth in the stored energy, where we want to study the influence of time-dependent boundary conditions. The analysis resembles that for temporally constant Dirichlet data with a time-dependent applied force carried out by the authors [9]. Yet, the argument is sketched to an extent that the differences become clear (for example, a restriction on the norm of the applied force required in [9] is no longer required here).

As mentioned in the Introduction, we work in a variational setting, where dislocation structures can be described by a nonconvex minimisation problem; see the pioneering work by Ortiz and Repetto [13] for a related setting for the analysis of plastically deformed crystals. The irreversibility is described by an incremental process (as suggested by [13]); a (phenomenological) dissipation functional is introduced to this behalf. As in the case of force-governed evolution [9], we consider an energy that depends on a strain tensor and has linear growth at infinity. The linear growth of the energy functional is necessitated by the plastic nature of the problem: it can be shown that in the setting of deformation theory of plasticity, the quasiconvex envelope of a single-slip energy has linear growth [2].

2.1. **The stored energy and its relaxation.** We first describe the energetic setting, casting it as a variational problem with a linear growth energy. The energy is assumed to be a continuous function $W\colon \bar\Omega \times \mathbb{R}^{n\times m} \to \mathbb{R}$ such that constants $\beta \geq \alpha > 0$ exist with

$$\alpha\,(|s|-1) \leq W(x,s) \leq \beta\,(1+|s|) \text{ for every } x \in \bar\Omega\ . \tag{3}$$

The motivation for the linear growth comes, as mentioned above, is natural in the setting of deformation theory of plasticity of single-slip systems, see [2].

The variational problem is then to

$$\text{minimise } I(u) := \int_\Omega W(x, \nabla u(x))\,\mathrm{d}x \text{ among } u \in W^{1,1}_{u_D}(\Omega;\mathbb{R}^m)\ . \tag{4}$$

In general, there is no solution to (4), because of the non-reflexivity of the underlying space and the possible non-(quasi)convexity of $W(x,\cdot)$. In order to capture the limiting behaviour of minimising sequences, we state a relaxed problem, still for a fixed instance of time (which we suppress from the notation for now). The relaxation is in terms of DiPerna-Majda measures $\eta = (\hat\nu, \sigma)$, see Appendix A. We write $\mathcal{GDM}^{u_D}_{\mathcal{F}}(\Omega;\mathbb{R}^{m\times n})$ for the set of DiPerna-Majda-measures generated by gradients of mappings in $\{u_k\}_{k\in\mathbb{N}} \subset W^{1,1}_{u_D}(\Omega;\mathbb{R}^m)$, with $u_D \in W^{1,1}(\Omega;\mathbb{R}^m)$ (see Appendix A). For the displacement u, the appropriate function space is $W^{1,\mu}\left(\bar\Omega;\mathbb{R}^m\right)$, the fine extension of $L^1(\Omega)$ in the sense of J. Souček [15], see Subsection 1.2. We note that it is *crucial* to work in this setting here, rather than in the more familiar setting of the space of bounded variations: $W^{1,\mu}\left(\bar\Omega;\mathbb{R}^m\right)$ gives *both* weak$\star$ compactness and weak$\star$ continuity of the trace, and this combination is essential for the problem under consideration. The relaxed formulation of (4) is

$$\text{minimise } \bar{I}\left(u, \bar{D}u, \sigma, \hat{\nu}\right) := \int_{\bar{\Omega}} \int_{\beta_{\mathcal{F}}\mathbb{R}^{m\times n}} \tilde{W}(x,s)\hat{\nu}_x(\mathrm{d}s)\,\sigma(\mathrm{d}x) \tag{5}$$

$$\text{among } \left(u, \bar{D}u\right) \in W^{1,\mu}\left(\Omega;\mathbb{R}^m\right) \text{ with } \bar{T}\left(u, \bar{D}u\right) = u_D \text{ on } \Gamma_D,$$

$$\text{and } (\sigma,\hat{\nu}) \in \mathcal{GDM}_{\mathcal{F}}^{u_D}(\Omega;\mathbb{R}^{m\times n}), \text{ where } \bar{D}u \text{ is given by}$$

$$\int_{\bar{\Omega}} \phi(x)\bar{D}u\,\mathrm{d}x = \int_{\bar{\Omega}} \phi(x) \int_{\beta_{\mathcal{F}}\mathbb{R}^{m\times n}} \frac{s}{1+|s|}\hat{\nu}_x(\mathrm{d}s)\,\sigma(\mathrm{d}x)\ . \tag{6}$$

It can be shown [9] that (5) has a solution and $\min \bar{I} = \inf I$, with I given in (4). Moreover, minimising sequences of I generate (in the sense of (44)) minimisers of $\bar{I}$ and every minimiser of $\bar{I}$ is generated by a minimising sequence of I.

Since microstructures can develop in the problem under consideration, it is reasonable to introduce a concept of a phase field variable, which we denote here λ. Motivated by applications in shape memory alloys, we introduce a variable akin to one used in [11]. We give one exact formulation below, but many variants are possible.

We suppose that there is $L \in \mathbb{N}$ and a continuous bounded mapping $\Lambda\colon \mathbb{R}^{m\times n} \to \mathbb{R}^L$ such that $\Lambda_j \in \mathcal{F}$ for $1 \le j \le L$ (with $\mathcal{F}$ a subalgebra of the space of bounded and continuous functions, see Appendix A) such that the mesoscopic order parameter λ associated with the system configuration described by $\left(u, \bar{D}u, \sigma, \hat{\nu}\right)$ is given by the formula

$$\lambda := \int_{\beta_{\mathcal{F}}\mathbb{R}^{m\times n}} \Lambda(s)\hat{\nu}_x(\mathrm{d}s)\sigma\ , \tag{7}$$

which means that $\lambda \in M\left(\bar{\Omega};\mathbb{R}^L\right)$ is a measure such that, for all $g \in C\left(\bar{\Omega}\right)$,

$$\int_{\bar{\Omega}} g(x)\lambda(\mathrm{d}x) = \int_{\bar{\Omega}} \int_{\beta_{\mathcal{F}}\mathbb{R}^{m\times n}} \Lambda(s)\hat{\nu}_x(\mathrm{d}s)g(x)\sigma(\mathrm{d}x)\ .$$

Here for $x \in \Omega$, ν_x is a probability measure; see Appendix A for the precise definition.

At present, it is common to augment the energy Γ by a regularising term to be able to prove existence. We follow this line of thought. We suppose that the measure $\lambda \in M\left(\bar{\Omega};\mathbb{R}^L\right)$ introduced in (7) is absolutely continuous with respect to the Lebesgue measure on Ω. We identify it with its density $x \mapsto \lambda(x)$. Moreover, we will require that λ, which is by definition integrable, belongs even to $W^{1,2}\left(\Omega;\mathbb{R}^L\right)$; see [11] for a similar regularisation, and a justification. Let $\varrho > 0$; we then consider

$$\Gamma_\varrho(t,q) := \int_{\bar{\Omega}\times\beta_{\mathcal{F}}\mathbb{R}^{m\times n}} \tilde{W}(x,s)\eta(\mathrm{d}s\mathrm{d}x) + \varrho\,\|\lambda(x)\|_{W^{1,2}(\Omega;\mathbb{R}^L)}\ . \tag{8}$$

Though the time-dependence may not be visible at first glance, Γ_ϱ depends on time since η is time-dependent. Finally, we set

$$\Gamma(t,q) = \begin{cases} \Gamma_\varrho(t,q) & \text{if } q \in Q \text{ and } \lambda \in W^{1,2}\left(\Omega;\mathbb{R}^L\right) \\ +\infty & \text{otherwise} \end{cases}, \tag{9}$$

with Q being the state space (defined rigorously in (10) below). Notice that (9) excludes states of the system in which λ is a measure which is not absolutely continuous with respect to the Lebesgue measure with fairly regular density. Existence of a minimiser for (9) follows from the existence argument given for (4) and the weak$\star$ compactness of the set of measures η which give rise to $\lambda \in W^{1,2}\left(\Omega;\mathbb{R}^L\right)$.

2.2. **Evolution.** We now describe the rate-independent evolution for a process with the energy (5), following the setting developed by Mielke and coworkers [12]. We consider the evolution during an arbitrary, but fixed time interval $[0, T]$. The evolution will be triggered by changes in the Dirichlet boundary data. To account for the energy that may be dissipated during the evolution, we follow Mielke and co-workers [11] in introducing a *dissipation distance*. As for the force-driven evolution [9], we define the (mesoscopic) dissipation distance between two DiPerna-Majda measures $\eta_1, \eta_2 \in GDM^{u_D}_{\mathcal{F}}(\Omega; \mathbb{R}^{m\times n})$, since these measures record the microstructure.

Let Q be the set of admissible configurations. Each such configuration will be written as $q := \big(u, \bar{D}u, \eta, \lambda\big)$. Since the boundary data depends on time, the set Q depends on time, but we decouple the time-dependence in the following way. At a given time t, let $u_D \in W^{1,\mu}(\Omega; \mathbb{R}^m)$ be the boundary data. Then, let η_D be the DiPerna-Majda measure generated by a subsequence of $\{\nabla u_k\}_{k\in\mathbb{N}} \subset L^1(\Omega; \mathbb{R}^m)$ from the definition of $W^{1,\mu}(\Omega; \mathbb{R}^m)$. Similarly, let λ_D be given by (7). We then seek a state $q \in Q$, where

$$Q := (u_D, \bar{D}u_D, \eta_D, \lambda_D) + Q_0 \ , \tag{10}$$

with Q_0 being is the set of admissible configurations with homogeneous Dirichlet data,

$$\begin{aligned} Q_0 := \big\{ q_0 = \big(u_0, \bar{D}u_0, \eta_0, \lambda_0\big) \ \text{with} \\ \big(u_0, \bar{D}u_0\big) \in W^{1,\mu}(\Omega; \mathbb{R}^m), \eta_0 \in GDM^{u_D}_{\mathcal{F}}\left(\Omega; \mathbb{R}^{m\times n}\right), \lambda_0 \in M\left(\bar{\Omega}; \mathbb{R}^L\right), \\ \bar{D}u_0 = \mathrm{Id} \bullet \eta_0, \ \lambda \text{ given by } (7), \text{ and } \bar{T}\left(u_0, \bar{D}u_0\right) = 0 \text{ on } \Gamma_D \ \big\}. \end{aligned}$$

Though Q depends on time, this is suppressed from the notation.

We now define the *dissipation* $\mathrm{D}\colon Q \times Q \to \mathbb{R}$ as

$$\mathrm{D}(q_1, q_2) = \|\lambda_1 - \lambda_2\|_{M(\bar{\Omega}; \mathbb{R}^L)} \ . \tag{11}$$

Since λ is derived from η, we sometimes write $\mathrm{D}(\eta_1, \eta_2)$ instead of $\mathrm{D}(q_1, q_2)$. We note that the time-dependent boundary conditions lead to a time-dependent DiPerna-Majda measure η and thus both λ and D vary over time. Also, as a consequence of (9), (11) can be written as $\mathrm{D}(\eta_1, \eta_2) = \|\lambda_1 - \lambda_2\|_{L^1(\Omega; \mathbb{R}^L)}$. We notice that D is symmetric, $\mathrm{D}(\eta_1, \eta_2) = \mathrm{D}(\eta_2, \eta_1)$ for every admissible pair (η_1, η_2). This condition is not essential and can be relaxed; see [1]. Also, the triangle inequality is valid for D. That is, for any three internal states η_1, η_2, η_3, it holds that

$$\mathrm{D}(\eta_1, \eta_3) \le \mathrm{D}(\eta_1, \eta_2) + \mathrm{D}(\eta_2, \eta_3) \ . \tag{12}$$

Finally, for a process $q\colon [0, T] \to Q$ and a given time interval $[t_1, t_2] \subset [0, T]$, the *temporal dissipation* is given by

$$\mathrm{Diss}(q, [t_1, t_2]) := \sup_{J\in\mathbb{N}} \left\{ \sum_{j=1}^{J} \mathrm{D}\left(q(\tau_{j-1}), q(\tau_j)\right) \ \Big| \ t_1 = \tau_0 < \cdots < \tau_J = t_2 \right\}.$$

We recall the definition of rate-independent processes as developed by Mielke and co-workers [12].

Definition 2.1. Given $q_0 \in Q$, we say that the process $q\colon [0, T] \to Q$ is a *solution* if the following conditions hold in addition to suitable regularity assumptions:

(i) *Global Stability:* For every $t \in [0,T]$, the process is stable in the global sense,

$$\Gamma(t,q(t)) \leq \Gamma(t,\tilde{q}) + \mathrm{D}(q(t),\tilde{q}) \text{ for every } \tilde{q} \in Q \ . \tag{13}$$

(ii) *Energy inequality:* For every $0 \leq t_1 \leq t_2 \leq T$, we have

$$\Gamma(t_1, q(t_1)) + \mathrm{Diss}(q,[t_1,t_2]) \leq \Gamma(t_2, q(t_2)) + \int_{t_1}^{t_2} \partial_t \Gamma(r, q(r))\,\mathrm{d}r \ , \tag{14}$$

(iii) *Initial condition:* $q(0) = q_0$ and $\Gamma(0,q(0)) < \infty$.

We need to define the notion of convergence in Q, and do so as follows.

Definition 2.2. Suppose that $\{q_k\}_{k\in\mathbb{N}} \subset Q$, where $q_k = \left(u_k, \bar{D}u_k, \eta_k, \lambda_k\right)$. We say that $q_k \rightharpoonup q := \left(u, \bar{D}u, \eta, \lambda\right) \in Q$ as $k \to \infty$ if $\left(u_k, \bar{D}u_k\right) \rightharpoonup \left(u, \bar{D}u\right)$ in $W^{1,\mu}(\Omega;\mathbb{R}^m)$, $\eta_k \overset{\star}{\rightharpoonup} \eta$ in $GDM_{\mathcal{F}}^{u_D}(\Omega;\mathbb{R}^{m\times n})$ and $\lambda_k \rightharpoonup \lambda$ in $W^{1,2}\left(\Omega;\mathbb{R}^L\right)$.

The main result of this paper is the following.

Theorem 2.3. *Under the assumptions stated in Subsection 3, there is a rate-independent process in the sense of Definition 2.1 with regularity*

$$\left(u, \bar{D}u\right) \in L^\infty\left(0,T;W^{1,\mu}(\Omega;\mathbb{R}^m)\right) \ , \tag{15}$$

$$\lambda \in BV\left(0,T;L^1\left(\Omega;\mathbb{R}^L\right)\right) \ . \tag{16}$$

Section 4 gives the proof of this theorem.

3. **Assumptions.** We recall the decomposition into time-dependent and homogeneous parts from (10), in particular $\eta = \eta_D + \eta_0$. Then Γ from (9) (respectively Γ_ρ from (8)) can be decomposed in a contribution with time-dependent boundary data and one with homogeneous Dirichlet data,

$$\Gamma_\varrho(t,q) := \int_{\bar{\Omega}\times\beta_{\mathcal{F}}\mathbb{R}^{m\times n}} \tilde{W}(x,s)\eta_D(\mathrm{d}s\mathrm{d}x) + \int_{\bar{\Omega}\times\beta_{\mathcal{F}}\mathbb{R}^{m\times n}} \tilde{W}(x,s)\eta_0(\mathrm{d}s\mathrm{d}x) + \varrho \left\|\lambda(x)\right\|_{W^{1,2}(\Omega;\mathbb{R}^L)}$$

(we don't split the regularising term here; the form above is sufficient to reveal the regularity we need).

For the time-dependent boundary data, we assume that

$$\left(u_D, \bar{D}u_D\right) \in C^1\left([0,T], W^{1,\mu}(\Omega;\mathbb{R}^m)\right) \ , \tag{17}$$

$$\eta_D \in C^1\left([0,T], GDM_{\mathcal{F}}^{u_D}\left(\Omega;\mathbb{R}^{m\times n}\right)\right) \ . \tag{18}$$

Also, we make the common assumption (see [5]) that there are constants $C_0, C_1 > 0$ such that

$$|\partial_t \Gamma(t,q)| \leq C_0(C_1 + \Gamma(t,q)) \ . \tag{19}$$

As a consequence we have

$$\Gamma(t_2,q) \leq (C_1 + \Gamma(t_1,q)) \exp\left(C_0 |t_2 - t_1|\right) - C_1 \ . \tag{20}$$

Further we require uniform continuity of $t \mapsto \partial_t\Gamma(t,q)$ in the sense that there is $\omega\colon [0,T] \to [0,+\infty)$ nondecreasing such that for all $t_1, t_2 \in [0,T]$

$$|\partial_t\Gamma(t_1,q) - \partial_t\Gamma(t_2,q)| \leq \omega\left(|t_1 - t_2|\right) \ . \tag{21}$$

We also suppose that $q \mapsto \partial_t\Gamma(t,q)$ is weakly continuous for all $t \in [0,T]$.

4. **Existence proof.** The existence of an energetic solution for a suitable $u_D \in C^1\left([0,T];W^{1,1}(\Omega;\mathbb{R}^m)\right)$ can be shown in a constructive way, using a sequence of incremental problems. We sketch the proof, which follows a now well-established argument, to highlight the incorporation of the time-dependent boundary conditions (see [7] for a similar argument in a different context, namely that of elastoplasticity). For a given initial condition $q_\tau^0 = q_0$ and a given step size τ, it is natural to define q_τ^k for $k = 1, \dots, N$ as a solution to the problem

$$\min_{q\in Q} \Gamma(k\tau, q) + \mathrm{D}\left(q_\tau^{k-1}, q\right). \tag{22}$$

We write for the time discretisation $0 = t_\tau^0 < \dots < t_\tau^N = T$ with $N = T/\tau \in \mathbb{N}$. The next proposition shows that $\{q_\tau^k\}_{k\in\mathbb{N}}$ is well-defined; accepting this for the moment, we introduce a piecewise interpolant q_τ such that $q_\tau(t) := q_\tau^{k-1}$ if $t \in [t_\tau^{k-1}, t_\tau^k)$ and $q_\tau(T) := q_\tau^N$.

Proposition 4.1. *The problem (22) has a solution q_τ^k which is stable; that is, for every $\tilde q \in Q$,*

$$\Gamma\left(k\tau, q_\tau^k\right) \le \Gamma(k\tau, \tilde q) + \mathrm{D}\left(q_\tau^k, \tilde q\right). \tag{23}$$

Moreover, for all $t_1 \le t_2$ from the set $\{k\tau\}_{k=0}^N$, the following discrete energy inequalities hold if one extends the definition of $q_\tau(t)$ by setting $q_\tau(t) := q_0$ if $t < 0$.

$$\begin{aligned}\int_{t_\tau^{k-1}}^{t_\tau^k} \partial_t\Gamma(s, q_\tau^k)\,\mathrm{d}s &\le \Gamma\left(t_\tau^k, q_\tau^k\right) + \mathrm{D}\left(q_\tau^{k-1}, q_\tau^k\right) - \Gamma\left(t_\tau^{k-1}, q_\tau^{k-1}\right)\\ &\le \int_{t_\tau^{k-1}}^{t_\tau^k} \partial_t\Gamma\left(s, q_\tau^{k-1}\right)\,\mathrm{d}s\ .\end{aligned} \tag{24}$$

The poof is now standard and thus omitted (see, e.g., [7] for details).

The next proposition gives the *a priori* bounds needed to pass to the limit as the step size goes to zero.

Proposition 4.2. *Assume that W satisfies the growth condition (3). Let further (19) and (21) hold.*

Then there is $\kappa \in \mathbb{R}$ such that

$$\left\|\left(u_\tau, \bar D u_\tau\right)\right\|_{L^\infty(0,T;W^{1,\mu}(\Omega;\mathbb{R}^m))} < \kappa\ , \tag{25}$$

$$\|\lambda_\tau\|_{L^\infty(0,T;W^{1,2}(\Omega;\mathbb{R}^L))\cap BV(0,T;L^1(\Omega;\mathbb{R}^L))} < \kappa\ , \tag{26}$$

$$\mathrm{Diss}(q_\tau, [0,T]) < \kappa\ , \tag{27}$$

and

$$\eta_\tau(\bar\Omega \times \beta_{\mathcal{F}}\mathbb{R}^{m\times n}) < \kappa\ . \tag{28}$$

Proof. Using (19), (21) and (24) we get the following *a priori* bounds for some constants $C_0, C_1 > 0$:

$$\Gamma(t_\tau^k, q_k) \le (\Gamma(0, q_0) + C_0)\exp\left(C_1 t_\tau^k\right) - C_1\ . \tag{29}$$

This, thanks to (3) gives us the bound (28). Hence, the first moments

$$\int_{\bar\Omega\times\beta_{\mathcal{F}}\mathbb{R}^{m\times n}} \frac{s}{1+|s|}\eta_\tau(\mathrm{d}s\mathrm{d}x)$$

of η_τ are bounded as well which, together with Lemma 1.1 shows (25). Similarly, we get (26). The estimate (27) follows from

$$\sum_{k=1}^{N} \mathrm{D}\left(q_\tau^k, q_\tau^{k-1}\right) \le C_0 \exp\left(C_1 T\right) \tag{30}$$

which holds independently of N and τ. □

Next, we define the set of stable states,

$$\mathcal{S}(t) := \left\{ q \in Q \;\middle|\; \Gamma(t,q) \le \Gamma\left(t,\tilde{q}\right) + \mathrm{D}\left(q,\tilde{q}\right) \text{ for every } \tilde{q} \in Q \right\} \;; \tag{31}$$

we recall that Q depends on time even if this is suppressed from the notation. We also define

$$\mathcal{S}_{[0,T]} := \cup_{t\in[0,T]} \{t\} \times \mathcal{S}(t) \;. \tag{32}$$

We say that a sequence $\{(t_k, q_k)\}_{k\in\mathbb{N}}$ is *stable* if $q_k \in \mathcal{S}\left(t_k\right)$.

The following proposition will help us to establish the stability of the limiting process.

Proposition 4.3. *Let Γ be weakly sequentially lower semicontinuous (as a function of q). Suppose that for all $(t_*, q_*) \in [0,T] \times Q$, for all stable sequences $\{(t_k, q_k)\}_{k\in\mathbb{N}}$ with $t_k \to t_*$ and $q_k \rightharpoonup q_*$ in the sense of Definition 2.2, there is a sequence $\{\tilde{q}_k\}_{k\in\mathbb{N}} \subset Q$ such that for all $\tilde{q} \in Q$*

$$\limsup_{k\to\infty} \left[\Gamma\left(t_k, \tilde{q}_k\right) + \mathrm{D}\left(q_k, \tilde{q}_k\right)\right] \le \Gamma\left(t_*, \tilde{q}\right) + \mathrm{D}\left(q_*, \tilde{q}\right) \;. \tag{33}$$

Then Γ is weakly continuous as a function of t and q along stable sequences and $q_* \in \mathcal{S}\left(t_*\right)$.

Proof. We adapt the proof of [10, Proposition 4.2] and first prove the weak continuity. Take $\tilde{q} := q_*$ in (33) and notice that by stability of q_k and then (33), we obtain for the choice $\tilde{q}_k := q_k$

$$\limsup_{k\to\infty} \Gamma\left(t_k, q_k\right) \le \limsup_{k\to\infty} \left[\Gamma\left(t_k, \tilde{q}_k\right) + \mathrm{D}\left(q_k, \tilde{q}_k\right)\right] \le \Gamma\left(t_*, \tilde{q}\right) + \mathrm{D}\left(q_*, \tilde{q}\right) = \Gamma\left(t_*, q_*\right) \;. \tag{34}$$

We have further

$$\lim_{k\to\infty} \left|\Gamma\left(t_k, q_k\right) - \Gamma\left(t_*, q_k\right)\right| = 0 \;,$$

due to assumption (18) (we recall that the time-dependence of W and hence Γ is due to the presence of η, see (8) and (9)). Since Γ is weakly lower semicontinuous it follows that

$$\liminf_{k\to\infty} \Gamma\left(t_k, q_k\right) = \liminf_{k\to\infty} \left[\Gamma\left(t_k, q_k\right) - \Gamma\left(t_*, q_k\right)\right] + \liminf_{k\to\infty} \Gamma\left(t_*, q_k\right) \ge \Gamma\left(t_*, q_*\right) \;.$$

This together with (34) gives weak continuity of $\Gamma\left(t_k, q_k\right) \to \Gamma\left(t_*, q_*\right)$. Finally, we show the stability of q_*. Using (33) we have for every $\tilde{q} \in Q$

$$\Gamma\left(t_*, q_*\right) = \lim_{k\to\infty} \Gamma\left(t_k, q_k\right) \le \limsup_{k\to\infty} \left[\Gamma\left(t_k, \tilde{q}_k\right) + \mathrm{D}\left(q_k, \tilde{q}_k\right)\right] \le \Gamma\left(t_*, \tilde{q}\right) + \mathrm{D}\left(q_*, \tilde{q}\right) \;.$$

The arbitrariness of $\tilde{q} \in Q$ shows the stability of q_*. □

Having the *a priori* estimates we follow [5] and use the Helly selection principle to find a subsequence of $\{\lambda_\tau\}$ (not relabeled) such that for all $t \in [0,T]$ $\lambda_\tau(t) \to \lambda(t)$ and $\lambda \in BV\left([0,T]; L^1\left(\Omega; \mathbb{R}^L\right)\right) \cap L^\infty\left(0,T; W^{1,2}\left(\Omega; \mathbb{R}^L\right)\right)$. Denoting $\theta_\tau(t) := \partial_t \Gamma(t, q_\tau(t))$ we have that $\{\theta_\tau\}_\tau$ is bounded in $L^\infty(0,T)$ by (18), so a subsequence (not relabelled) converges weakly$\star$ to a limit θ. Moreover,

$$\theta(t) \le \theta_{\mathrm{s}}(t) := \limsup_{\tau\to 0} \theta_\tau(t)$$

by Fatou's lemma. We choose t-dependent sequences such that

$$(u_\tau(t), \bar{D}u_\tau(t)) \to (u(t), \bar{D}u(t)) \text{ weakly}\star \text{ in } W^{1,\mu}(\bar{\Omega};\mathbb{R}^m) \ ,$$

$$\theta_\tau(t) \to \theta_{\rm s}(t) \ ,$$

and

$$\eta_\tau(t) \to \eta(t) \text{ weakly}\star \text{ in } M(\bar{\Omega}\times\beta_{\mathcal{F}}\mathbb{R}^{m\times n}) \ .$$

We can also suppose that $\delta(t) := \lim_{\tau\to 0} \mathrm{Diss}(q_\tau;[0,t])$ exists as it is the limit of a bounded nondecreasing sequence.

We now consider the situation where $0 \le t-k\tau \le \tau$; then $q_\tau(t) = q_\tau(k\tau)$. Hence, using (24) in the first two lines and exploiting that q_τ is piecewise constant, we find that for some $C, C_1 > 0$

$$\begin{aligned}\Gamma\left(t,q_\tau(t)\right) + \mathrm{Diss}\left(q_\tau;[0,t]\right) &\le \Gamma\left(k\tau,q_\tau(k\tau)\right) + \mathrm{Diss}\left(q_\tau;[0,k\tau]\right) + C\tau \\ &\le \Gamma\left(0,q_\tau(0)\right) - \int_0^{k\tau} \partial_t\Gamma\left(s,q_\tau(s)\right)\,\mathrm{d}s + C\tau \\ &\le \Gamma\left(0,q_\tau(0)\right) - \int_0^{t} \partial_t\Gamma\left(s,q_\tau(s)\right)\,\mathrm{d}s + C_1\tau \ .\end{aligned}$$

We now proceed as in [7]. We define the pointwise infimum $\theta_{\rm i}(t) := \liminf_{\tau\to 0}\theta_\tau(t)$. Further, using Helly's Theorem we get in the limit $\tau \to 0$

$$\Gamma(t,q(t)) + \delta(t) \le \Gamma(0,q(0)) - \int_0^t \theta(s)\,\mathrm{d}s \ . \tag{35}$$

As $\delta(t) \ge \mathrm{Diss}\left(q;[0,t]\right)$ by the weak lower semicontinuity of the dissipation, and by Fatou's lemma $\int_0^t \theta(s)\,\mathrm{d}s \ge \int_0^t \theta_{\rm i}(s)\,\mathrm{d}s$ for a.e. $t\in[0,T]$, we obtain

$$\Gamma(t,q(t)) + \mathrm{Diss}\left(q;[0,t]\right) \le \Gamma(0,q(0)) - \int_0^t \theta_{\rm i}(s)\,\mathrm{d}s \ .$$

We observe that $\theta_{\rm i}(s) = \partial_t\Gamma\left(s,q(s)\right)$. Altogether we get the upper energy estimate

$$\Gamma(t,q(t)) + \mathrm{Diss}\left(q;[0,t]\right) \le \Gamma(0,q(0)) - \int_0^t \partial_t\Gamma(s,q(s))\,\mathrm{d}s \ . \tag{36}$$

In order to get the lower estimate we exploit the fact that $q(t)$ is stable for all $t\in[0,T]$. Take a (possibly non-uniform) partition of a time interval $[t_1,t_2]\subset[0,T]$ such that $t_1 = \vartheta_0 < \vartheta_1 < \vartheta_2 < \vartheta_K = t_2$ such that $\max_i\left(\vartheta_i - \vartheta_{i-1}\right) =: \vartheta \to 0$ as $K\to\infty$. We test the stability of $q\left(\vartheta_{k-1}\right)$ with $q\left(\vartheta_k\right)$, for $k = k_1+1,\ldots,k_2$. After a summation over k, this yields

$$\begin{aligned}&-\sum_{k=1}^{K}\left[\Gamma\left(\vartheta_{k-1},q\left(\vartheta_k\right)\right) - \Gamma\left(\vartheta_k,q\left(\vartheta_k\right)\right)\right] \\ &\qquad\qquad \le \Gamma\left(t_2,q\left(t_2\right)\right) - \Gamma\left(t_1,q\left(t_1\right)\right) + \sum_{k=1}^{K}\mathrm{D}\left(q\left(\vartheta_{k-1}\right),q\left(\vartheta_k\right)\right) \ ,\end{aligned} \tag{37}$$

which immediately implies

$$\sum_{k=1}^{K}\int_{\vartheta_{k-1}}^{\vartheta_k}\partial_t\Gamma\left(s,q\left(\vartheta_k\right)\right)\,\mathrm{d}s \le \Gamma\left(t_2,q\left(t_2\right)\right) - \Gamma\left(t_1,q\left(t_1\right)\right) + \mathrm{Diss}\left(q;[t_1,t_2]\right) \ . \tag{38}$$

We re-write the expression on the left as

$$\sum_{k=1}^{K}\int_{\vartheta_{k-1}}^{\vartheta_k}\partial_t\Gamma\left(s,q\left(\vartheta_k\right)\right)\,\mathrm{d}s=\sum_{k=1}^{K}\partial_t\Gamma\left(\vartheta_k,q\left(\vartheta_k\right)\right)\left(\vartheta_k-\vartheta_{k-1}\right)$$
$$+\sum_{k=1}^{K}\int_{\vartheta_{k-1}}^{\vartheta_k}\left[\partial_t\Gamma\left(s,q\left(\vartheta_k\right)\right)-\partial_t\Gamma\left(\vartheta_k,q\left(\vartheta_k\right)\right)\right]\,\mathrm{d}s\ . \quad (39)$$

The second term on the right-hand side of (39) tends to zero as $\vartheta\to 0$ because the time derivative of the measure η is uniformly continuous in time by (18). The first term on the right-hand side converges to $\int_{t_1}^{t_2}\partial_t\Gamma(s,q(s))\,\mathrm{d}s$ (see [3, Lemma 4.12] for the argument). Thus, (38) and (39) together yield the lower energy bound

$$\int_{t_1}^{t_2}\partial_t\Gamma(s,q(s))\,\mathrm{d}s\le\Gamma\left(t_2,q\left(t_2\right)\right)-\Gamma\left(t_1,q\left(t_1\right)\right)+\mathrm{Diss}\left(q;\left[t_1,t_2\right]\right)\ . \quad (40)$$

The upper and lower estimates (36) and (40) together yield the claimed energy balance

$$\Gamma(t,q(t))+\mathrm{Diss}\left(q;[0,t]\right)=\Gamma(0,q(0))-\int_0^t\partial_t\Gamma(s,q(s))\,\mathrm{d}s\ . \quad (41)$$

Step 3: With a now established argument, it follows that

$$\Gamma(0,q(0))-\int_0^t\theta_{\mathrm{i}}(s)\,\mathrm{d}s\le\Gamma(0,q(0))-\int_0^t\theta(s)\,\mathrm{d}s\le\Gamma(0,q(0))-\int_0^t\theta_{\mathrm{i}}(s)\,\mathrm{d}s \quad (42)$$

(see, e.g., [7]). □

Stability of q follows from Proposition 4.3. Altogether we shown that an energetic solution exists, which finishes the proof of Theorem 2.3.

Appendix A. **DiPerna-Majda measures.** In some situations, oscillations and concentration phenomena can occur. Oscillation effects can be described with Young measures, which describe the limit of a sequence $\{u_k\}_{k\in\mathbb{N}}$ of functions $u_k\colon\Omega\to\mathbb{R}^d$ converging weakly in $L^p\left(\Omega;\mathbb{R}^d\right)$ for $1\le p<\infty$ respectively weakly$\star$ for $p=\infty$. A *Young measure* on Ω is a mapping with values in the probability measures, $\Omega\to\mathrm{Prob}\left(\mathbb{R}^d\right)$, $x\mapsto\nu_x$ which is *weak$\star$ measurable*; this means that for every $x\in\Omega$ and any every $f\in C_0\left(\mathbb{R}^d\right)$, the mapping

$$\Omega\to\mathbb{R},\qquad x\mapsto\langle f,\nu_x\rangle:=\int_{\mathbb{R}^d}f\left(s\right)\nu_x\left(\mathrm{d}s\right)$$

is measurable in the usual sense.

DiPerna-Majda are an extension of this concept for situations where additional concentration effects can occur. This happens as a consequence of the lack of a bound in $L^p\left(\Omega;\mathbb{R}^d\right)$ with $1<p\le\infty$. One is then often left to work in $L^1\left(\Omega;\mathbb{R}^d\right)$; then concentration effects may occur due to the non-reflexivity.

We now describe DiPerna-Majda measures, following the discussion in [9] but specialising the discussion to linear growth ($p=1$). The classic introduction to these measures is the original paper [4]. Let f be a function $\mathbb{R}^d\to\mathbb{R}$ with linear growth at infinity. DiPerna-Majda measures then describe the limit of a sequence $\{f\left(u_k\right)\}_{k\in\mathbb{N}}$, where the functions $u_k\colon\Omega\to\mathbb{R}^d$ converge weakly in $L^1\left(\Omega;\mathbb{R}^d\right)$ but are not uniformly bounded in $L^\infty\left(\Omega;\mathbb{R}^d\right)$.

More generally, DiPerna-Majda measures can be defined for Polish spaces (that is, a topological space X whose topology can be induced by a distance d that makes

(X, d) complete and separable). While an open X set of $\mathbb{R}^n$ (with respect to the Euclidean metric) can be equipped with a (non-Euclidean) metric so that X becomes Polish, we work here on the closure of the given set so that the time-dependent boundary effects can be studied in detail. (This is in the spirit of Subsection 1.2, where it is also convenient to include concentrations on the boundary.) As in [9], we thus present a slight modification of DiPerna's and Majda's result by considering $\bar{\Omega}$ (and test functions $\phi \in C(\bar{\Omega})$) rather than open domains Ω as in [4]; see [14, Subsection 3.2c] for the same modification. This change only amounts to replacing the isomorphism between the dual space of $(C_0(\Omega), \|\cdot\|)$ and the space $(M(\Omega), \|\cdot\|)$ of Radon measures with finite mass by the isomorphism of $\left(C(\bar{\Omega}), \|\cdot\|\right)$ and the space of Radon measures with compact support $\left(M(\bar{\Omega}), \|\cdot\|\right)$.

The definition of DiPerna-Majda measures involves a compactification; this is discussed in greater detail in [9, Appendix A]. As described there, we examine a completely regular subalgebra $\mathcal{F}$ of the space of bounded continuous functions $BC\left(\mathbb{R}^d\right)$. As an example, one can consider the compactification $\beta_{\mathcal{F}}\mathbb{R}^d$ by a sphere. In this case, $\mathcal{F}$ contains all functions $\tilde{f}$ for which the radial limit $\lim_{r\to\infty} \tilde{f}(rs)$ exists for arbitrary $s \in \mathbb{R}^d$ (but $\mathcal{F}$ may also contain functions $\tilde{f}$ without well-defined radial limits). To deal with functions f with linear growth at infinity in a convenient manner, we set $\tilde{f}(s) := \frac{f(s)}{1+|s|}$, with $\tilde{f} \in \mathcal{F}$.

Suppose we are given a sequence $\{u_k\}_{k\in\mathbb{N}}$, uniformly bounded in $L^1\left(\Omega, \mathbb{R}^d\right)$, and seek to describe the weak limit

$$\lim_{k\to\infty} \int_\Omega \phi(x) f(u_k(x))\, \mathrm{d}x \ ,$$

with $\phi \in C_0(\Omega)$ and $f(s) = \tilde{f}(s)(1+|s|)$, where $\tilde{f} \in \mathcal{F} \subset BC\left(\mathbb{R}^d\right)$ as above. A canonical norm for f of this form is $|f|_\infty := \max_{s\in\mathbb{R}^d} \tilde{f}(s) = \left|\tilde{f}\right|_\infty$.

DiPerna and Majda have shown [4, Theorem 4.1] that for a bounded sequence $\{u_k\}_{k\in\mathbb{N}}$ in $L^1\left(\bar{\Omega}; \mathbb{R}^d\right)$, there exists a non-negative Radon measure $\sigma \in M^+\left(\bar{\Omega}\right)$ such that

$$(1 + |u_k(x)|)\, \mathrm{d}x \overset{\star}{\rightharpoonup} \sigma \text{ in } M\left(\bar{\Omega}\right) \ . \tag{43}$$

Furthermore, for a separable completely regular subalgebra $\mathcal{F}$ of $BC\left(\mathbb{R}^d\right)$, there exist a σ-measurable map $\hat{\nu}\colon \Omega \to \operatorname{Prob}\left(\beta_{\mathcal{F}}\mathbb{R}^d\right)$, $x \mapsto \hat{\nu}_x$, and a subsequence of $\{u_k\}_{k\in\mathbb{N}}$ (not relabelled) such that for every $\tilde{f} \in \mathcal{F}$

$$\lim_{k\to\infty} \int_{\bar{\Omega}} \phi(x) f(u_k(x))\, \mathrm{d}x = \int_{\bar{\Omega}} \phi(x) \int_{\beta_{\mathcal{F}}\mathbb{R}^d} \tilde{f}(s)\, \hat{\nu}_x(\mathrm{d}s)\, \sigma(\mathrm{d}x) \tag{44}$$

holds for every $\phi \in C\left(\bar{\Omega}\right)$ [4, Theorem 4.3]. We say that $\{u_k\}_{k\in\mathbb{N}}$ *generates* the pair $(\sigma, \hat{\nu})$ if (44) holds. A pair $(\sigma, \hat{\nu}) \in M^+\left(\bar{\Omega}\right) \times L^\infty_{\mathrm{w}}\left(\bar{\Omega}, \sigma; \operatorname{Prob}\left(\beta_{\mathcal{F}}\mathbb{R}^d\right)\right)$ attainable by sequences in $L^1\left(\Omega; \mathbb{R}^d\right)$ is called a *DiPerna-Majda measure* (here $L^\infty_{\mathrm{w}}\left(\bar{\Omega}, \sigma; \operatorname{Prob}\left(\beta_{\mathcal{F}}\mathbb{R}^d\right)\right)$ is the dual space of $L^1\left(\bar{\Omega}, \sigma; C\left(\beta_{\mathcal{F}}\mathbb{R}^d\right)\right)$). The set of all DiPerna-Majda measures is denoted $\mathcal{DM}_{\mathcal{F}}\left(\Omega; \mathbb{R}^d\right)$. The explicit description of the set of DiPerna-Majda measures $\mathcal{DM}_{\mathcal{F}}\left(\Omega; \mathbb{R}^d\right)$ for unconstrained sequences is given in [6, Theorem 2].

In the bulk of this article, we use an alternative description of DiPerna-Majda measures. Specifically, in analogy to the proof of Theorem 4.1 in [4], we consider measures η in $M\left(\bar{\Omega} \times \beta_{\mathcal{F}}\mathbb{R}^d\right)$ and say that $\{u_k\}_{k\in\mathbb{N}} \subset L^1\left(\Omega; \mathbb{R}^d\right)$ *generates* the measure $\eta \in M\left(\bar{\Omega} \times \beta_{\mathcal{F}}\mathbb{R}^d\right)$ if, for every $\tilde{h} \in C\left(\bar{\Omega} \times \beta_{\mathcal{F}}\mathbb{R}^d\right)$,

$$\lim_{k\to\infty} \int_{\bar{\Omega}} \tilde{h}\left(x, u_k\left(x\right)\right)\left(1 + \left|u_k\left(x\right)\right|\right) \mathrm{d}x = \int_{\bar{\Omega}\times\beta_{\mathcal{F}}\mathbb{R}^d} \tilde{h}\left(x, s\right)\eta\left(\mathrm{d}s\mathrm{d}x\right) \tag{45}$$

holds. We write $DM_{\mathcal{F}}\left(\Omega;\mathbb{R}^d\right)$ denote the set of all measures generated in this way. The dictionary linking this definition with the one given above (and thus the justification for calling $DM_{\mathcal{F}}\left(\Omega;\mathbb{R}^d\right)$ DiPerna-Majda measures) is as follows. Since $\phi\left(x\right)\tilde{f}\left(y\right)$ with $\phi \in C\left(\bar{\Omega}\right)$ and $\tilde{f} \in C\left(\beta_{\mathcal{F}}\mathbb{R}^d\right)$ is dense in $C\left(\bar{\Omega} \times \beta_{\mathcal{F}}\mathbb{R}^d\right)$, it is natural to say that $\eta \cong (\sigma, \hat{\nu})$ for $\eta \in DM_{\mathcal{F}}\left(\Omega;\mathbb{R}^d\right)$ and a DiPerna-Majda measure $(\sigma, \hat{\nu})$ if

$$\langle \eta, \tilde{h}\rangle := \int_{\bar{\Omega}\times\beta_{\mathcal{F}}\mathbb{R}^d} \tilde{h}\left(x, s\right)\eta\left(\mathrm{d}x\mathrm{d}s\right) = \int_{\bar{\Omega}}\int_{\beta_{\mathcal{F}}\mathbb{R}^d} \tilde{h}\left(x, s\right)\hat{\nu}_x\left(\mathrm{d}s\right)\,\sigma\left(\mathrm{d}x\right)$$

for any $\tilde{h} \in C\left(\bar{\Omega} \times \beta_{\mathcal{F}}\mathbb{R}^d\right)$.

It is known [14, Chapter 3] that $DM_{\mathcal{F}}\left(\Omega;\mathbb{R}^d\right)$ is a closed, convex, non-compact but locally compact and locally sequentially compact subset of the locally convex space $M\left(\bar{\Omega} \times \beta_{\mathcal{F}}\mathbb{R}^d\right)$ in the weak$\star$ topology.

In summary, we view DiPerna-Majda measures as finite Radon measures on $\bar{\Omega} \times \beta_{\mathcal{F}}\mathbb{R}^d$, where $\bar{\Omega} \times \beta_{\mathcal{F}}\mathbb{R}^d$ is equipped with the Borel σ-algebra. The topology of DiPerna-Majda measures is that of the weak-$\star$ topology (see Subsection 1.1). We remark that a for suitable space Ω, for probability measures defined on Ω or more for generally measures which are uniformly bounded in norm, the weak-$\star$ topology is metrisable [16, Theorem 1.1.2]. However, DiPerna-Majda measures are not necessarily uniformly bounded.

We denote by $\mathcal{GDM}_{\mathcal{F}}\left(\Omega;\mathbb{R}^{m\times n}\right)$ the subset of $\mathcal{DM}_{\mathcal{F}}\left(\Omega;\mathbb{R}^{m\times n}\right)$ of those measures which are generated by gradients of mappings in $W^{1,1}\left(\Omega;\mathbb{R}^m\right)$. Expressed differently, $(\sigma, \hat{\nu}) \in \mathcal{GDM}_{\mathcal{F}}\left(\Omega;\mathbb{R}^{m\times n}\right)$ if there is $\{u_k\}_{k\in\mathbb{N}} \subset W^{1,1}\left(\Omega;\mathbb{R}^m\right)$ such that for all $\phi \in C\left(\bar{\Omega}\right)$ and all $\tilde{f} \in \mathcal{F}$

$$\lim_{k\to\infty} \int_{\bar{\Omega}} \phi\left(x\right) f\left(\nabla u_k\left(x\right)\right)\mathrm{d}x = \int_{\bar{\Omega}}\int_{\beta_{\mathcal{F}}\mathbb{R}^{m\times n}} \phi\left(x\right)\tilde{f}\left(s\right)\hat{\nu}_x\left(\mathrm{d}s\right)\sigma\left(\mathrm{d}x\right)\ . \tag{46}$$

Similarly we write $\eta \in GDM_{\mathcal{F}}\left(\Omega;\mathbb{R}^{m\times n}\right)$ if $\eta \in DM_{\mathcal{F}}\left(\Omega;\mathbb{R}^{m\times n}\right)$ is generated by gradients. Finally, $\mathcal{GDM}^{u_D}_{\mathcal{F}}\left(\Omega;\mathbb{R}^{m\times n}\right)$ denotes elements $(\sigma, \hat{\nu}) \in \mathcal{GDM}_{\mathcal{F}}\left(\Omega;\mathbb{R}^{m\times n}\right)$ with the property that $(\sigma, \hat{\nu})$ is generated by $\{u_k\}_{k\in\mathbb{N}} \subset W^{1,1}_{u_D}\left(\Omega;\mathbb{R}^m\right)$, with $u_D \in W^{1,1}\left(\Omega;\mathbb{R}^m\right)$.

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